Vortices and chirality of magnetostatic modes in quasi-2D ferrite disc particles

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Abstract
Microwave vortices in ferrite particles can appear in different kinds of physical phenomena depending on space scales of the wave processes. In this paper, we show that the vortex states can be created not only in magnetically soft 'small' (with the dipolar and exchange energy competition) cylindrical dots, but also in magnetically saturated 'big' (when the exchange fluctuations are neglected) cylindrical dots. A property associated with a vortex structure becomes evident from an analysis of confinement phenomena of magnetic oscillations in a ferrite disc with a dominating role of magnetic-dipolar (non-exchange-interaction) spectra. In this case, the scalar (magnetostatic-potential) wavefunctions may have a phase singularity in the centre of a dot. A non-zero azimuth component of the flow velocity demonstrates the vortex structure. The vortices are guaranteed by the chiral edge states of magnetic-dipolar modes in a quasi-2D ferrite disc.

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1. Introduction

It is well known that when a system consisting of many interacting particles is set rotating, it may form vortices. Vortices can be easily observed in water. They are a common occurrence in plasma and atmospheric science. In recent years, there has been a considerable interest in vortices of electromagnetic fields in vacuum. There are electromagnetic processes with broken symmetry characterized by different topological effects. The recognition that vortices are rather important manifestations of optical phase singularities has opened up a new frontier in optics [1–3]. Recent studies in optics show that for a case of a nanoparticle illuminated by the electromagnetic field, the 'energy sink' vortices with spiral energy flow line trajectories are seen in the proximity of the nanoparticle's plasmon resonance [4]. It becomes apparent that in microwaves, ferrite particles with the ferromagnetic-resonance (FMR) conditions should provoke creation of electromagnetic vortices. In papers [5, 6], ferrite inclusions with the FMR conditions were used to analyse a role of the time-reversal symmetry breaking as a

factor leading to creation of the Poynting-vector microwave vortices. There are vortices of the electromagnetic fields due to material (ferrite) gyrotropy and sample geometry.

Not only electromagnetic vortices are created by ferrite samples in microwaves. In the proximity of the FMR one may have vortices of intrinsic magnetic oscillations in a ferrite sample. It is well known that in ferrites there can be different magnetic wave processes characterized by short-range (exchange interaction) and long-range (dipole–dipole interaction) space scales [7, 8]. It turns out that the vortex structures in magnetically soft 'small' ferrite dots with the dipolar and exchange energy competition and in 'big' magnetically saturated particles with a dominating dipole energy contribution have different physical nature.

Magnetically soft ferromagnetic materials generally form domain structures to reduce their magnetostatic (MS) energy. In this context, closure domains are especially suitable. Such magnetic objects are characterized by a closed flux circuit having no magnetic flux leakage outside the material. In very small systems, however, the formation of domain walls is not energetically favoured. Specifically, in a dot of a ferromagnetic material of micrometre or submicrometre size, a curling spin configuration—that is, a magnetization vortex—has been proposed to occur in place of domains. The vortex consists of an in-plane, flux-closure magnetization distribution and a central core whose magnetization is perpendicular to the dot plane. It has been shown that under certain conditions a vortex structure will be stable because of competition between the exchange and dipole interactions. A magnetic vortex means the 'curling magnetization distribution'. One obtains the clockwise and counterclockwise rotations of magnetization vector $\vec{m}$ in the dot plane [9–11].

In a general case, both short-range exchange and long-range dipole–dipole (magnetostatic) interactions contribute to eigenfrequencies of the collective spin excitation. The importance of the MS energy increases gradually as the particle size increases. MS ferromagnetism has a character essentially different from exchange ferromagnetism [7, 8, 12, 13]. This statement finds strong confirmation in confinement phenomena of magnetic-dipolar-mode (MDM) oscillations. The dipole interaction provides us with a long-range mechanism of interaction, where a magnetic medium is considered as a continuum. Contrary to an exchange spin wave, in magnetic-dipolar waves the local fluctuation of magnetization does not propagate due to the interaction between the neighbouring spins. When field differences across the sample become comparable to the bulk demagnetizing fields the local-oscillator approximation is no longer valid, and indeed under certain circumstances entirely new spin dynamics behaviour can be observed. This dynamics behaviour is the following. Precession of magnetization about a vector of a bias magnetic field produces a small oscillating magnetization $\vec{m}$ and a resulting dynamic demagnetizing field $\vec{H}$, which reacts back on the precession, raising the resonant frequency. In the continuum approximation, vectors $\vec{H}$ and $\vec{m}$ are coupled by the differential relation:

$$\vec{\nabla} \cdot \vec{H} = -4\pi \vec{\nabla} \cdot \vec{m}, \tag{1}$$

where

$$\vec{H} = -\nabla \psi \tag{2}$$

and ψ is a MS potential. This, together with the Landau–Lifshitz (LL) equation, leads to complicated integro-differential equations for the mode solutions in a lossless magnetic sample.

For calculation, the formulation based on the MS-Green-function integral problem for magnetization $\vec{m}$ was suggested and used in [14–16]. In this case, one solves 'pure static' MS equations for the dipolar field. It is supposed that the sources of a MS field are both volume and surface 'magnetic charges' arising from $\nabla \cdot \vec{m}$ and from discontinuity of the normal component

of $\vec{m}$ on the surface of a ferrite sample. The MS potential ψ is defined based on integration of the MS Poisson equation

$$\nabla^2 \psi = -4\pi \rho_m, \tag{3}$$

where

$$\rho_m \equiv -\vec{\nabla} \cdot \vec{m}. \tag{4}$$

For the edge and volume sources, one obtains [17]

$$\psi = -\int_V \frac{\vec{\nabla}_{r'} \cdot \vec{m}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, \mathrm{d}V + \oint_S \frac{\vec{n}' \cdot \vec{m}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, \mathrm{d}S. \tag{5}$$

Such a theoretical analysis of the radio-frequency (RF) magnetization eigenvalue problem encounters, however, a significant difficulty due to the absence of exact information of the boundary conditions for RF magnetization $\vec{m}$. It is well known that in classical electrodynamics the boundary conditions are imposed on the normal components of the magnetic induction and the tangential component of the magnetic fields, but not on the components of magnetization. So the dynamic magnetization at the boundary of a magnetic element is undefined from classical electrodynamics. To derive dipolar boundary conditions for dynamic magnetization, one calculates the MS energy arising from the effective magnetic charge at the magnetic element. The obtained phenomenological pinning parameter for RF magnetization results from the induced surface 'magnetic charges' at the edges of a finite-size non-elipsoidal magnetic element. The physical meaning of this pinning parameter is to minimize the surface MS energy. To solve the eigenvalue problem for magnetic-dipolar magnetization one considers the inhomogeneous dipolar field obtained directly from magnetostatic equation (5) and uses the vector torque equation for magnetization [14–16]. The solution of the problem yields a discrete set of eigenfunctions. It has been concluded that these eigenfunctions of magnetization are orthogonal with real eigenvalues because the kernel of the Fredholm-type integral equation is symmetric and real.

An analysis gives a good correlation with the experimental results for 'small' (submicron size) magnetic particles. At the same time, the approach cannot be considered generally as well suited to describe spectral characteristics of MDMs in a 'big' magnetic particle. It is known that in a case of spectral problems, for integral equation with positive-definite symmetrical kernel there should be a corresponding Hermitian differential operator (see, e.g., [18]). Now the question arises: what are the Hermitian-operator differential equations corresponding to the integral MS operators in [14–16]? Certainly, the Poisson equation (3) is not a Hermitian-operator differential equation. Also there are no corresponding Hermitian-operator differential equations written for the RF MDM (non-exchange-interaction) magnetization $\vec{m}$. One cannot use the short-range exchange differential operator to solve the far-range dipolar boundary problem. So, in the MDM problem one cannot expand the RF magnetization by the eigenfunctions of the exchange operator. The MS spectral problem, being analysed based on the above approach, is not the self-conjugate problem and statements about completeness of the magnetization eigenfunctions [14–16] may cast certain doubts. The dipolar-contribution factor becomes also a very complicated problem for the calculation of the magnetic normal modes based on micromagnetic simulations. In this case, the dipolar energy is considered as the most difficult part to treat, compared to other types of the magnetic energy. The Poisson-equation approach is not well suited to yield the derivatives needed for the dynamical magnetization torque matrix [19].

The flaw of the above integral eigenvalue problem becomes evident for 'big' (with sizes tens of micrometres and more) ferrite samples. In the continuum approximation, to get correct solutions for dynamical processes inside a 'big' ferrite particle one has to presuppose the

existence of certain retardation effects for the MS fields. For 'big' magnetic samples, an analysis of the eigenvalue problem based on the differential-operator equation for a fictitious MS-potential wavefunction ψ may eliminate the above difficulties of the MS-Green-function integral problem. In this case, one uses the continuum approximation based on the known (from the linearized local (non-exchange-interaction) LL equation) permeability tensor $\ddot{\mu}$ [8]. One supposes that there is a spectral problem for the MS-potential propagating fields which cause and govern propagation of magnetization fluctuations. In other words, spacetime magnetization fluctuations are corollaries of the propagating MS-potential fields, but there is no the magnetization-wave spectral problem. The boundary conditions are imposed on the MS-potential field and not on the RF magnetization[1]. Usually, to calculate these effects the Walker's [7, 8] differential formulation is used and the general solution of this equation is expressed through the MS-potential wavefunction ψ:

$$\vec{\nabla} \cdot (\ddot{\mu} \cdot \vec{\nabla}\psi) = 0. \tag{6}$$

Equation (6) follows immediately from equations (1) and (2). The way of solution based on the Walker's equation is used both for continuous-wave ferromagnetic-resonance [7, 8] and nuclear-magnetic-resonance [20] measurements. The boundary conditions one obtains immediately from an analysis of the spectral problem for MS-potential wavefunction ψ.

In this paper, we show that the vortex states can be created not only in magnetically soft 'small' (with the dipolar and exchange energy competition) cylindrical dots, but also in magnetically saturated 'big' (when the exchange fluctuations are neglected) cylindrical dots. Our definition of the long-range order is based on the overall properties of the system rather than on the behaviour of a two-point correlation function. A property associated with a vortex structure becomes evident from an analysis of the spectral problem for MS-potential wavefunction ψ. We show that in a case of a normally magnetized thin-film ferrite disc, MDM vortices can be guaranteed by chirality of the MS waves. The dynamical symmetry breaking in MS-wave oscillations shows special gauge transformation of the fields.

In spite of the fact that vortices can appear in different kinds of physical phenomena, yet such 'swirling' entities seem to elude an all-inclusive definition. A magnetization vortex in a magnetically soft sample cannot be characterized by some invariant, such as the flux of vorticity. So, a vorticity thread may not be defined for the magnetization vortex [21]. At the same time, the MDM vortices take place due to a non-zero azimuth component of the flow velocity. It is well known that in quantum systems the gradient of the scalar potential plays the role analogous to the velocity. The circulation of the velocity therefore becomes quantized. Our analysis demonstrates such quantum-like vortices in normally magnetized MDM ferrite discs.

2. Energy relations for MDMs in a normally magnetized ferrite disc

It was supposed that expression for power flow density for MS waves can be derived with formal use of the Maxwell equations for the curl electric field and the potential magnetic field [22]. This may give, in particular, a foundation for well-known theories of the MS-wave excitation by an electric current (see, e.g., [23, 24]). Based on this expression, Boardman *et al* [25] considered the power-flow-density rotation excited by three planar antennas to

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[1] This is similar to the physics of the electromagnetic wave process in an ordinary transmission-line system. In this case the electromagnetic wave propagation causes space-time fluctuation of a conductivity current in metal parts and an electric displacement current in dielectrics of a transmission line, but there are no spectral problems for the electric-charge-density waves. So, the boundary conditions are imposed on the electromagnetic field components and not on the RF currents.

analyse possible vortex structures of MS waves in a ferrite film. It looks, however, that this way of an analysis may be used as a proper approximation for the main MS mode but reveals a clear physical contradiction for a general MS-wave spectral problem. Concerning the MS-wave propagation effects, it was disputed in [26, 27] that from a classical electrodynamics point of view one does not have a physical mechanism describing the effect of transformation of the curl electric field to the potential magnetic field. Also the gauge transformation in this derivation does not fall under the known gauge transformations, neither the Lorentz gauge nor the Coulomb gauge, and cannot formally lead to the wave equation [27].

MS waves can propagate only due to ferrite-medium confinement phenomena [8, 22]. So, the power flow density of MS waves should be considered via an analysis of the mode spectral problem in a certain waveguide structure, but not based on an analysis of the wave propagation in a boundless magnetic medium. It means that the MS-wave power flow density should have the only physical meaning as a norm of a certain propagating mode in a magnetic waveguide structure. As a necessary consequence, this leads to the question of orthogonality and completeness of MS modes in such a waveguide. In this problem, MS-potential ψ acquires a special physical meaning as a scalar wavefunction in a Hilbert functional space. Such a standpoint is especially important for the study of the MS-wave vortices: in confined magnetic structures, vortices should be characterized as stable and energy favoured 'swirling' entities.

For MS waves one can use the following operator equation for MS potential ψ and magnetic flux density $\vec{B}$ [28]:

$$\hat{L} V = 0, \tag{7}$$

where $\hat{L} \equiv \begin{pmatrix} (\overleftrightarrow{\mu})^{-1} & \nabla \\ -\nabla\cdot & 0 \end{pmatrix}$ is the differential-matrix operator and $V \equiv \begin{pmatrix} \vec{B} \\ \psi \end{pmatrix}$. When MS waves propagate in an infinite ferrite rod along the z-axis, equation (7) can be rewritten as

$$(\hat{L}_\perp - \mathrm{i}\beta_z \hat{R}) \tilde{V} = 0, \tag{8}$$

where $\hat{L}_\perp \equiv \begin{pmatrix} (\overleftrightarrow{\mu})^{-1} & \nabla_\perp \\ -\nabla_\perp\cdot & 0 \end{pmatrix}$ (subscript $\perp$ means differentiation over a waveguide cross section), β_z is the MS-wave propagation constant along the z-axis, $\tilde{V} \equiv \begin{pmatrix} \vec{\tilde{B}} \\ \tilde{\psi} \end{pmatrix}$ is the membrane vector function ($V \equiv \tilde{V} \mathrm{e}^{-\mathrm{i}\beta_z z}$), $\hat{R} \equiv \begin{pmatrix} 0 & \vec{e}_z \\ -\vec{e}_z & 0 \end{pmatrix}$, $\vec{e}_z$ is a unit vector along the axis of the wave propagation. Outside of a ferrite region one has the same equations but with $\overleftrightarrow{\mu} = \overleftrightarrow{I}$, where $\overleftrightarrow{I}$ is the unit matrix [28].

Integration by parts on S—a square of an open MS-wave cylindrical waveguide—of the integral $\int_S (\hat{L}_\perp \tilde{V}) \tilde{V}^* \, \mathrm{d}S$ gives the contour integral in a form $\oint_C (\tilde{B}_r \tilde{\psi}^* - \tilde{B}_r^* \tilde{\psi}) \, \mathrm{d}C$, where C is a contour surrounding a cylindrical ferrite core and $\tilde{B}_r$ is a radial component of a membrane function of the magnetic flux density. Operator $\hat{L}_\perp$ becomes self-adjoint for homogeneous boundary conditions (continuity of $\tilde{\psi}$ and $\tilde{B}_r$) on contour C. Based on such homogeneous boundary conditions one obtains the orthogonality relation for MDMs:

$$(\beta_p - \beta_q) \int_S (\hat{R} \tilde{V}_p)(\tilde{V}_q)^* \, \mathrm{d}S = 0. \tag{9}$$

It is easy to show that norm $N_p = \int_S (\tilde{\psi}_p \vec{\tilde{B}}_p^* - \tilde{\psi}_p^* \vec{\tilde{B}}_p) \cdot \vec{e}_z \, \mathrm{d}S$, being multiplied by a proper dimensional coefficient, corresponds to the power flow of the waveguide mode p through a waveguide cross section. For monochromatic fields with time variation $\sim \mathrm{e}^{\mathrm{i}\omega t}$ we have for the power flow in Gaussian units

$$P_p = -\frac{\mathrm{i}\omega}{16\pi} N_p = \frac{\mathrm{i}\omega}{16\pi} \int_S (\tilde{\psi}_p^* \vec{\tilde{B}}_p - \tilde{\psi}_p \vec{\tilde{B}}_p^*) \cdot \vec{e}_z \, \mathrm{d}S. \tag{10}$$

The proof of this fact is evident. Based on equation (7) (together with the equation complex conjugated with equation (7)) and taking into account expression $\vec{B} = -\vec{\vec{\mu}}(\omega)\nabla\psi$, one obtains

$$-\frac{\mathrm{i}\omega}{16\pi}\nabla\cdot(\psi\vec{B}^* - \psi^*\vec{B}) = -\frac{\mathrm{i}\omega}{16\pi}[\vec{B}\cdot(\vec{\vec{\mu}}^*(\omega))^{-1}\cdot\vec{B}^* - \vec{B}^*\cdot(\vec{\vec{\mu}}(\omega))^{-1}\cdot\vec{B}]. \tag{11}$$

This is an energy balance equation for monochromatic MS waves in lossy magnetic media. On the right-hand side of this equation we have the average density of magnetic losses taken with an opposite sign. Thus, the term on the left-hand side is the divergence of the power flow density. Really, in a region of a FMR, the average density of magnetic energy absorption can be expressed as [8, 29]

$$\left\langle\frac{\partial w}{\partial t}\right\rangle_{\text{abs}} = \frac{\mathrm{i}\omega}{8\pi}\vec{H}^*\cdot\vec{\vec{\mu}}^{\text{ah}}(\omega)\cdot\vec{H}, \tag{12}$$

where $\vec{H}$ is an internal radio-frequency (RF) magnetic field and superscript ah means 'anti-Hermitian'. Since $\vec{B}\cdot(\vec{\vec{\mu}}^*(\omega))^{-1}\cdot\vec{B}^* - \vec{B}^*\cdot(\vec{\vec{\mu}}(\omega))^{-1}\cdot\vec{B} = 2\vec{H}^*\cdot\vec{\vec{\mu}}^{\text{ah}}(\omega)\cdot\vec{H}$, the right-hand side of equation (11) describes the density of magnetic losses taken with an opposite sign.

We introduced magnetic losses in the above analysis to be able to define clearly the power flow density of the MS-wave waveguide mode. The next question is about the average density of accumulated energy for MS-wave modes. The fact that components of permeability tensor $\vec{\vec{\mu}}$ are dependent on both frequency ω and bias magnetic field H_0 may lead to unique features in spectral problems of MS oscillations. For given mode p in a ferrite rod axially magnetized along the z-axis one rewrites equation (6) as follows [28]:

$$\left(\hat{G}_\perp - \left(\beta_{z_p}\right)^2\right)\tilde{\psi}_p = 0, \tag{13}$$

where $\hat{G}_\perp \equiv \mu\nabla_\perp^2$, $\nabla_\perp^2$ is the two-dimensional (with respect to cross-sectional coordinates) Laplace operator, μ is a diagonal component of tensor $\vec{\vec{\mu}}$ [8]. Operator $\hat{G}_\perp$ is *positive definite* for negative quantities μ. Outside a ferrite region equation (13) becomes the Laplace equation ($\mu = 1$). Double integration by parts on square S—a cross-section of a waveguide structure—of the integral $\int_S (\hat{G}_\perp\tilde{\psi})\tilde{\psi}^*\,\mathrm{d}S$ gives the boundary conditions for self-adjointness of operator $\hat{G}_\perp$. For given frequency ω in a region of a bias magnetic field H_0 where $\mu < 0$, one has a complete orthonormal basis of eigen modes $\tilde{\psi}$ of operator $\hat{G}_\perp$. Eigen numbers are quantities $\left(\beta_{z_p}\right)^2$ [27, 28].

Let us represent membrane function $\tilde{\psi}_p$ as

$$\tilde{\psi}_p = C_p\tilde{\varphi}_p, \tag{14}$$

where C_p is a dimensional normalization coefficient and $\tilde{\varphi}_p$ is a dimensionless membrane function for mode p [30]. We also formally introduce a certain mode quantity E_p which is defined as

$$E_p \equiv \frac{g_p}{16\pi}\left(\beta_{z_p}\right)^2. \tag{15}$$

Here g_p is another dimensional normalization coefficient for mode p. The physical meaning of quantities E_p and g will be shown and discussed later on. Based on relation (15) and taking into account equation (14) one rewrites equation (13) in a form

$$\hat{F}_\perp\tilde{\varphi}_p = E_p\tilde{\varphi}_p, \tag{16}$$

where $\hat{F}_\perp \equiv \frac{g_p}{16\pi}\mu\nabla_\perp^2$. Assuming self-adjointness of operator $\hat{G}_\perp$ (and, consequently, self-adjointness of operator $\hat{F}_\perp$) one obtains the following orthonormality conditions:

$$(E_p - E_q)\int_S \tilde{\varphi}_p\tilde{\varphi}_q^*\,\mathrm{d}S = 0. \tag{17}$$

It follows that $\int_S \tilde{\varphi}_p \tilde{\varphi}_q^* \, \mathrm{d}S = \delta_{pq}$, where δ_{pq} is the Kronecker delta. Based on the Walker equation for an axially magnetized ferrite rod $\mu \nabla_{\perp}^2 \psi + \nabla_{\parallel}^2 \psi = 0$, where subscript $\parallel$ means differentiation along the z-axis, one rewrites equation (16) as

$$\hat{F}_{\parallel} \tilde{\varphi}_p = -E_p \tilde{\varphi}_p. \tag{18}$$

Here $\hat{F}_{\parallel} \equiv \frac{g_p}{16\pi} \nabla_{\parallel}^2$. For given mode p, equation (18) looks like the time-independent one-dimensional Schrödinger equation for a free particle [31]. For two modes, p and q, one obtains

$$(E_p - E_q)\tilde{\varphi}_p \tilde{\varphi}_q^* = \nabla_{\parallel} \cdot (\tilde{\varphi}_p \nabla_{\parallel} \tilde{\varphi}_q^* - \tilde{\varphi}_q^* \nabla_{\parallel} \tilde{\varphi}_p). \tag{19}$$

Integration of this equation over cross-section S gives immediately the orthogonality condition (17). Based on formal resemblance with the quantum mechanics description, MS-mode propagation in an infinite ferrite rod can be considered as a 'free-particle motion' with 'potential energy' constant along the z-axis.

Now let us consider a ferrite disc normally magnetized along the z-axis. For a ferrite disc with thickness d, one has restrictions of a 'potential energy' in the z-direction. At $z = 0, d$ there are the boundary conditions: continuity of $\tilde{\varphi}$ and $\nabla_{\parallel} \tilde{\varphi}$ [28, 30]. For given mode p, there is a specific quantity of E_p and a specific height of an effective confining potential U_p^{eff}. In a case of a ferrite disc, equation (18) should be represented as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{16\pi}{g_p} E_p \right) \tilde{\varphi}_p = 0 \tag{20}$$

for $0 \leqslant z \leqslant d$ and

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{16\pi}{g_p} \left(E_p - U_p^{\mathrm{eff}} \right) \right] \tilde{\varphi}_p = 0 \tag{21}$$

for $z \leqslant 0; z \geqslant d$. Inside a ferrite disc ($0 \leqslant z \leqslant d$) one has $\left(\beta_{z_p} \right)^2 = \frac{16\pi}{g_p} E_p$, while outside a disc ($z \leqslant 0; z \geqslant d$), for $E_p < U_p^{\mathrm{eff}}$, one has [30] $\tilde{\varphi}_p = \mathrm{const} \cdot \mathrm{e}^{\mp \alpha_p z}$, where $(\alpha_p)^2 = \frac{16\pi}{g_p} \left(U_p^{\mathrm{eff}} - E_p \right)$. Here signs $\mp$ correspond to regions $z \geqslant d$ and $z \leqslant 0$, respectively. It follows (see [28]) that $\alpha = \frac{1}{\sqrt{-\mu}} \beta$, where μ is a negative quantity.

After some transformations, one obtains from the above expressions

$$\frac{U^{\mathrm{eff}}}{E} = 1 + \frac{1}{-\mu}. \tag{22}$$

For a FMR region with negative μ [8], a character of dependence $\frac{U^{\mathrm{eff}}}{E} = \frac{U^{\mathrm{eff}}}{E}(H_0)$, where H_0 is a bias magnetic field, is shown in figure 1. In this figure, the boundary quantities of a bias magnetic field are [8] $H_{0_1} = \frac{\omega}{\gamma}$; $H_{0_2} = \left[\left(\frac{\omega}{\gamma} \right)^2 + 2\pi M_0 \right]^{1/2} - 2\pi M_0$, where γ is the gyromagnetic ratio and M_0 is the saturation magnetization. Based on the known positions of MDM resonances in a ferrite disc with respect to H_0 [30], one can fit proper mode quantities of $\frac{U^{\mathrm{eff}}}{E}$. Bias magnetic fields corresponding to different MDMs (p, q, r) are designated in figure 1 as $H_0^{(p)}, H_0^{(q)}, H_0^{(r)}$. For $p < q < r$, there are $H_0^{(p)} > H_0^{(q)} > H_0^{(r)}$. Qualitative pictures of the U^{eff} and E levels for three modes p, q, r in 'potential wells' are shown in figure 2. One has $U_p^{\mathrm{eff}} < U_q^{\mathrm{eff}} < U_r^{\mathrm{eff}}$ and $E_p < E_q < E_r$.

Further comparison with quantum mechanics approach leads to understanding real physical meaning of the above formal analysis. From the principle of superposition of states, it follows that wavefunctions $\tilde{\varphi}_p$ $(p = 1, 2, \ldots)$, describing our 'quantum' system, are 'vectors' in an abstract space of an infinite number of dimensions—the Hilbert space. In quantum mechanics, this is the case of the so-called *energetic representation*, when the system
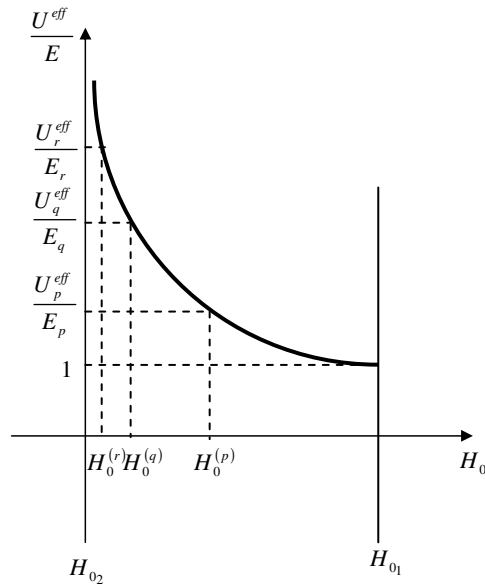
Figure 1. Dependence of quantity $\frac{U^{\text{eff}}}{E}$ on a bias magnetic field for magnetic-dipolar oscillating modes in a ferrite disc.
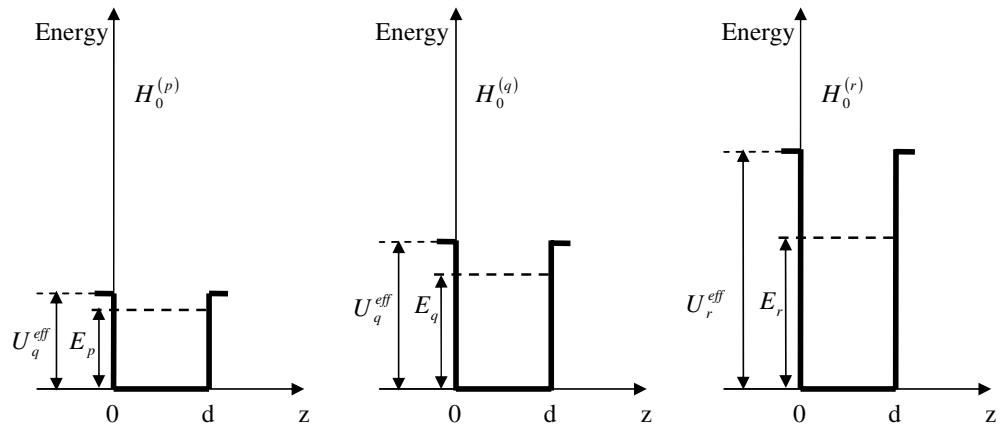
Figure 2. A model illustrating the levels of the 'kinetic' and potential energies for magnetic-dipolar oscillating modes in a ferrite disc.

energy runs through a discrete sequence of values. In the energetic representation, a square of a modulus of the wavefunction defines probability to find a system with a certain energy value [31, 32]. In our case, scalar-wave membrane function $\tilde{\varphi}$ can be represented as $\tilde{\varphi} = \sum_p a_p \tilde{\varphi}_p$ and the probability to find a system in a certain state p is defined as $|a_p|^2 = \left| \int_S \tilde{\varphi} \tilde{\varphi}_p^* \, \mathrm{d}S \right|^2$. The complete-set orthonormalized basic vectors in Hilbert space describe a stationary 'ensemble'. In an infinite axially magnetized ferrite rod one can consider the case of the evolution of the 'ensemble'. The MS-potential wavefunction ψ entirely defines the state of our system—the magnetic sample. It means that representation of this function in a certain time moment not

only describes the system behaviour at the present moment, but defines the behaviour in all future time moments. Mathematically, it signifies that (taking into account the principle of superposition) there should be [31]

$$i\frac{\partial \psi}{\partial t} = \hat{Q}\psi, \tag{23}$$

where $\hat{Q}$ is a certain linear operator. From this equation it can be shown that for orthonormalized basic vectors operator $\hat{Q}$ is a self-conjugate differential operator. Thus, equation (23) is a wave equation for complex scalar wavefunction ψ [31].

Let us represent a MS-potential function of mode p propagating in an infinite axially magnetized ferrite rod as a quasi-monochromatic quantity:

$$\Psi_p(z,t) = \psi_p^{(\mathrm{max})}(z,t)\,\mathrm{e}^{\mathrm{i}(\omega t - \beta_p z)}, \tag{24}$$

where 'quasi-membrane' complex amplitude $\psi_p^{(\mathrm{max})}(z,t)$ is a smooth function of the longitudinal coordinate and time, so that $\left|\frac{1}{\beta_{z_p}}\frac{\partial \psi_p^{(\mathrm{max})}}{\partial z}\right| \ll \psi_p^{(\mathrm{max})}$ and $\left|\frac{1}{\omega}\frac{\partial \psi_p^{(\mathrm{max})}}{\partial t}\right| \ll \psi_p^{(\mathrm{max})}$. Situation of the quasi-monochromatic behaviour can be realized, in particular, by means of a time-dependent bias magnetic field slowly varying with respect to the Larmor frequency. In this case, a spin-polarized ensemble will adiabatically follow the bias magnetic field and the resulting energy of interaction with a bias field becomes time dependent.

We represent now the quasi-monochromatic MS field described by equation (24) as

$$\Psi_p(z,t) = C_p\Phi_p(z,t), \tag{25}$$

where C_p is a dimensional normalization coefficient (the same as in equation (14)) and $\Phi_p(z,t)$ is a dimensionless MS-potential function. Equation (23) can be written as

$$i\frac{\partial \Phi_p}{\partial t} = \frac{\omega}{\left(\beta_{z_p}\right)^2}\nabla_\parallel^2\Phi_p. \tag{26}$$

The form of operator $\hat{Q} \equiv \frac{\omega}{\left(\beta_{z_p}\right)^2}\nabla_\parallel^2$ follows from the 'stationary-state' conditions. Taking into account equation (15), one sees that equation (26) has a physical meaning as a quasi-monochromatic generalization of the 'stationary-state' equation (18) describing MS modes propagating in a ferrite rod. The 'stationary-states' correspond to monochromatic fields with time variation $\sim\mathrm{e}^{\mathrm{i}\omega t}$. Based on equation (26) and its complex conjugated form one easily obtains

$$\frac{\partial |\Phi_p|^2}{\partial t} = \frac{\mathrm{i}\omega}{\left(\beta_{z_p}\right)^2}\nabla_\parallel \cdot (\Phi_p\nabla_\parallel\Phi_p^* - \Phi_p^*\nabla_\parallel\Phi_p). \tag{27}$$

For a quasi-monochromatic field, the energy balance equation for waveguide mode p propagating along the z-axis in a lossless ferrite rod is written as

$$\int_z \mathrm{d}z \int_S \frac{\partial \langle w_p\rangle_{\mathrm{acc}}}{\partial t}\,\mathrm{d}S = -\int_z \mathrm{d}z \int_S \nabla_\parallel \cdot \vec{p}_p\,\mathrm{d}S, \tag{28}$$

where $\langle w_p\rangle_{\mathrm{acc}}$ is the average density of accumulated magnetic energy of MS mode p and $\vec{p}_p$ is the power flow density of mode p. Based on equations (10) and taking into account expression for the permeability tensor [8], we have for an axially magnetized ferrite rod

$$\frac{\partial \langle w_p\rangle_{\mathrm{acc}}}{\partial t} = \frac{\mathrm{i}\omega}{16\pi}\nabla_\parallel \cdot (\Psi_p\nabla_\parallel\Psi_p^* - \Psi_p^*\nabla_\parallel\Psi_p). \tag{29}$$

Comparison of equations (27) and (29), with taking into account equation (25), gives

$$\langle w_p\rangle_{\mathrm{acc}} = \frac{C_p^2\left(\beta_{z_p}\right)^2}{16\pi}|\Phi_p|^2. \tag{30}$$

Based on equation (15) and orthogonality conditions for membrane functions $\tilde{\varphi}$, one has

$$\int_S \langle w_p \rangle_{\mathrm{acc}} \, \mathrm{d}S = \frac{C_p^2}{g_p} \int_S E_p \, \mathrm{d}S. \tag{31}$$

Condition $g_p \equiv C_p^2$ leads us to important conclusion about a physical meaning of quantities E_p and g_p introduced formally above, in equation (15). The quantity g_p should be considered as a dimensional normalization coefficient with the same dimension as squared MS-mode-amplitude coefficient C_p. The quantity E_p means an average density of accumulated magnetic energy of MS mode p propagating in a ferrite rod.

From quantum mechanics, it is known that based on the one-dimensional time-dependent Schrödinger equation, $i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \nabla_\parallel^2 \phi + U\phi$, one has the continuity equation $\frac{\partial |\phi^2|}{\partial t} = -\frac{i\hbar}{2m} \nabla_\parallel \cdot (\phi \nabla_\parallel \phi^* - \phi^* \nabla_\parallel \phi)$. One can see a clear correlation between this continuity equation and the scalar-wavefunction equation (27) when coefficient $\frac{\omega}{(\beta_{z_p})^2}$ is formally replaced by $-\frac{\hbar}{2m}$. This also corresponds to the de Broglie dispersion relation for a particle of effective mass m_p and moment $\hbar \beta_{z_p}$:

$$\omega = \frac{\hbar}{2m_p} (\beta_{z_p})^2, \tag{32}$$

where the effective mass m_p is a *negative* quantity. This relation shows that there should be the negative frequency ω. The physical meaning of such 'negativeness' follows from the negative-dispersion character of MS modes propagating in an axially magnetized ferrite rod [33].

One may wonder whether the MDMs are quantum or classical. We cannot answer this question in a simple question, but give some remarks on it as follows. In our case, we have evident both classical and quantum attributes. In the Maxwell theory, we can define a positive-definite energy density, while cannot define a positive-definite probability density. The above analysis shows that every MS mode propagating in a ferrite rod and every MS mode oscillating in a ferrite disc can be characterized by probability density and are described by the Schrödinger-like equation. Propagation of MS modes can be considered as propagation of quasiparticles—the light magnons (LMs). The meaning of the term 'light magnon' arises from the fact that the quantities of effective masses of these quasiparticles are much less than effective masses of the ('real', 'heavy') magnons—the quasiparticles existing due to the exchange interaction [34]. In our description of MS oscillations we neglect the exchange-interaction fluctuations and the dynamical 'magnetic stiffness' is characterized by the 'weak' dipole–dipole interaction. Expression (32) looks very similar to an effective mass of the 'heavy' magnon for exchange-interaction spin waves with the quadratic character of dispersion [8]. The above analysis shows the way of transition from the classical theory to the quantum-like theory of MDMs in a ferrite sample. When one selects properly the normalization of a linear combination of differential-equation solutions for MS-potential functions, one may attain coincidence between MS field energy in a sample and energy of an oscillator system. LMs are long-range elementary magnetic excitations which appear due to dipole–dipole interactions in confined-system magnetically ordered continuum. Every oscillating MDM in a ferrite sample is characterized by a certain type of the LMs. There are different LM effective masses for different oscillating MDMs. So in a finite ferrite sample, LMs related to different MDMs (or, in other words, related to different energy states) are not identical particles [34].

3. Self-adjointness of differential operators and boundary conditions

In the above consideration, we supposed self-adjointness of both differential operators, $\hat{G}_\perp$ and $\hat{L}_\perp$. Nevertheless, boundary conditions for self-adjointness of operator $\hat{G}_\perp$ differ from boundary conditions necessary for self-adjointness of operator $\hat{L}_\perp$. The boundary conditions for self-adjointness of operator $\hat{G}_\perp$ is defined based on double integration by parts on square S—a cross-section of a waveguide structure—of the integral $\int_S (\hat{G}_\perp \tilde{\psi}) \tilde{\psi}^* \, \mathrm{d}S$. The boundary condition for self-adjointness of operator $\hat{G}_\perp$ can be written as [27, 30]

$$\mu (H_r)_{r=\Re^-} - (H_r)_{r=\Re^+} = 0, \tag{33}$$

where H_r is a radial component of the RF magnetic field. At the same time, as we pointed out above, operator $\hat{L}_\perp$ becomes self-adjoint for homogeneous boundary conditions (continuity of $\tilde{\psi}$ and $\tilde{B}_r$) on a lateral surface. The homogeneous boundary condition for the radial component of $\vec{B}$ in a cylindrical ferrite rod of radius $\Re$ is written as

$$\mu (H_r)_{r=\Re^-} - (H_r)_{r=\Re^+} = -\mathrm{i}\mu_a (H_\theta)_{r=\Re}. \tag{34}$$

The quantity $(H_\theta)_{r=\Re}$ means the azimuth magnetic field on the border circle. For magnetostatic solutions it becomes clear [27] that because of the boundary condition (34) membrane functions cannot be considered as single-valued functions. This fact raises a question about the validity of the energy orthogonality relation for MS-wave modes. For a system for which a total Hamiltonian is conserved, there should be single valuedness for eigenfunctions. Thus, we are arguing that the internal disc region (where diagonal component of the permeability tensor $\mu < 0$) must have a domain wall at the edge. One can use a notion of a domain wall as a model of the edge. As we will show, in order to cancel the 'edge anomaly' the boundary excitation must be described by chiral states.

Following a standard way of solving boundary problems in mathematical physics [35, 36], one can consider two joint boundary problems: the main boundary problem and the conjugate boundary problem. The problems are described by differential equations which are similar to equation (8). The main problem is expressed by a differential equation $(\hat{L}_\perp - \mathrm{i}\beta \hat{R})\tilde{V} = 0$. The conjugate problem is expressed by an equation $(\hat{L}_\perp^\circ - \mathrm{i}\beta^\circ \hat{R})\tilde{V}^\circ = 0$. From a formal point of view, it is supposed initially that these are different equations: there are different differential operators, different eigenfunctions and different eigenvalues. A form of differential operator $\hat{L}_\perp^\circ$ is obtained from integration by parts:

$$\int_S (\hat{L}_\perp \tilde{V})(\tilde{V}^\circ)^* \, \mathrm{d}S = \int_S \tilde{V}(\hat{L}_\perp^\circ \tilde{V}^\circ)^* \, \mathrm{d}S + \oint_C P(\tilde{V}, \tilde{V}^\circ) \, \mathrm{d}C, \tag{35}$$

where C is a contour surrounding a cylindrical ferrite core and $P(\tilde{V}, \tilde{V}^\circ)$ is a bilinear form.

Operator $\hat{L}_\perp$ is a self-conjugate operator when permeability tensor $\ddot{\mu}$ is a Hermitian tensor and when a contour integral $\oint_C P(\tilde{V}, \tilde{V}^\circ) \, \mathrm{d}C$ is equal to zero. We can represent the contour integral on the right-hand side of equation (35) as [27]

$$\oint_C P(\tilde{V}, \tilde{V}^\circ) \, \mathrm{d}C = \oint_C \left[\mu \left(\frac{\partial \tilde{\varphi}}{\partial r} \right)_{r=\Re^-} - \left(\frac{\partial \tilde{\varphi}}{\partial r} \right)_{r=\Re^+} \right] (\tilde{\varphi}^\circ)^*_{r=\Re}$$

$$- (\tilde{\varphi})_{r=\Re} \left[\mu \left(\frac{\partial \tilde{\varphi}^\circ}{\partial r} \right)_{r=\Re^-} - \left(\frac{\partial \tilde{\varphi}^\circ}{\partial r} \right)_{r=\Re^+} \right]^* \, \mathrm{d}C$$

$$+ \oint_C \left[\left(\mathrm{i}\mu_a \frac{\partial \tilde{\varphi}}{\partial \theta} \right) (\tilde{\varphi}^\circ)^* - (\tilde{\varphi}) \left(\left(\mathrm{i}\mu_a \frac{\partial \tilde{\varphi}}{\partial \theta} \right)^\circ \right)^* \right]_{r=\Re} \, \mathrm{d}C. \tag{36}$$

The dimensionless membrane function $\tilde{\varphi}$ is considered as a function satisfying boundary conditions (34). Evidently, this function changes a sign when θ is rotated by 2π. So there are

two solutions: $\tilde{\varphi}_+$ and $\tilde{\varphi}_-$. We introduce now the dimensionless membrane function $\tilde{\eta}$ which satisfies boundary condition (33). Since $\hat{G}_\perp$ and $\hat{L}_\perp$ are linear operators, we can write the following relation:

$$\tilde{\eta}(\rho, \alpha) = \begin{cases} \gamma_- \tilde{\varphi}_+ \\ \gamma_+ \tilde{\varphi}_-, \end{cases} \tag{37}$$

where $\gamma_\mp \equiv a_\mp \, \mathrm{e}^{-iq_\mp \theta}$ is an auxiliary function. To preserve the single-valued nature of the membrane function, function $\gamma_\mp$ must change its sign when θ is rotated by 2π so that $\mathrm{e}^{-iq_\mp 2\pi} = -1$. That is $q_\mp = \mp l \frac{1}{2}$, where $l = 1, 3, 5, \ldots$. We rewrite equation (37) as follows:

$$\tilde{\varphi}_\pm = \frac{1}{\gamma_\mp} \tilde{\eta} = \delta_\pm \tilde{\eta}, \tag{38}$$

where $\delta_\pm = \frac{1}{a_\mp} \mathrm{e}^{-iq_\pm \theta} \equiv f_\pm \mathrm{e}^{-iq_\pm \theta}$. The quantity $q_\pm$ is equal to $\pm l \frac{1}{2}$. In the above relations, evidently, $a_+ = -a_-$ and $f_+ = -f_-$. To have proper normalization we will take $|a_\mp| = |f_\pm| = 1$.

Functions of the conjugate problem have the forms similar to those written for functions of the main problem:

$$\tilde{\varphi}_\pm^\circ = \frac{1}{\gamma_\mp^\circ} \tilde{\eta}^\circ = \delta_\pm^\circ \tilde{\eta}^\circ, \tag{39}$$

where $\delta_\pm^\circ \equiv f_\pm^\circ \mathrm{e}^{-iq_\pm^\circ \theta}$, $f_+^\circ = -f_-^\circ$, $|f_\pm^\circ| = 1$ and $q_\pm^\circ$ are half-integer numbers.

The edge states are defined by four edge functions: δ_+, δ_-, δ_+°, and δ_-°. Based on a simple analysis (see [27]) one can reduce equation (36) to

$$\oint_C P_\pm(\tilde{V}, \tilde{V}^\circ) \, \mathrm{d}C = \Re \int_0^{2\pi} \left[\left(i\mu_a \frac{\partial \delta_\pm}{\partial \theta} \right) (\delta_\pm^\circ)^* - (\delta_\pm) \left(\left(i\mu_a \frac{\partial \delta_\pm}{\partial \theta} \right)^\circ \right)^* \right]_{r=\Re} \mathrm{d}\theta. \tag{40}$$

Here, putting signs for a bilinear form $P(\tilde{V}, \tilde{V}^\circ)$, we took into account the fact that there is a possibility for two (or positive (+) or negative (−)) solutions on the boundary.

A sign of a full chiral rotation, $q_+ \theta = \pi$ or $q_- \theta = -\pi$, should be correlated with a sign of the parameter $i\mu_a$. This becomes evident from the fact that a sign of $i\mu_a$ is related to a precession direction of a magnetic moment $\vec{m}$. In a ferromagnetic resonance, the bias field sets up a preferential precession direction. For a normally magnetized ferrite disc with the bias field directed along the z-axis, a transverse component of a magnetic moment $\vec{m}$ precesses counterclockwise about the field. Because of the preferential precession direction for a given direction of a bias field, the following condition should be used in our analysis:

$$(i\mu_a)^\circ = i\mu_a. \tag{41}$$

So, one obtains

$$\oint_C P_\pm(\tilde{V}, \tilde{V}^\circ) \, \mathrm{d}C = \Re \int_0^{2\pi} \left[\left(i\mu_a \frac{\partial \delta_\pm}{\partial \theta} \right) (\delta_\pm^\circ)^* - (\delta_\pm) \left(i\mu_a \frac{\partial \delta_\pm^\circ}{\partial \theta} \right)^* \right]_{r=\Re} \mathrm{d}\theta$$
$$= \Re \mu_a \left(q_\pm - q_\pm^\circ \right) \int_0^{2\pi} [\delta_\pm (\delta_\pm^\circ)^*]_{r=\Re} \, \mathrm{d}\theta. \tag{42}$$

Figure 3 illustrates two cases of directions of a chiral rotation in a correlation with directions of the RF magnetization evolution in a ferrite disc. We will call these cases as the (+) resonance (when a direction of a chiral rotation coincides with the precession magnetization direction) and the (−) resonance (when a direction of a chiral rotation is opposite to the precession magnetization direction). The MS-wave membrane function maps to itself under simultaneous change of a sign of the magnetization vector and shift of the phase of function
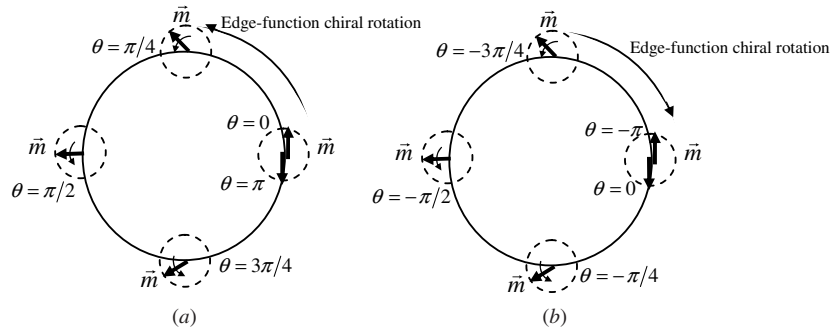
Figure 3. Directions of an edge-function chiral rotation in a correlation with directions of the RF magnetization $\vec{m}$ evolution in a ferrite disc. (*a*) The (+) resonance and (*b*) the (−) resonance.

δ by $+\pi$ (in a case of the (+) resonance) or by $-\pi$ (in a case of the (−) resonance). It means that for function δ both the counterclockwise and clockwise directions of going around make physical sense.

From demand of self-adjointness of operator $\hat{L}_\perp$ one has the orthogonality conditions $(q_+ - q_+^\circ) \int_0^{2\pi} [\delta_+ (\delta_+^\circ)^*]_{r=\Re} \, d\theta = 0$ and $(q_- - q_-^\circ) \int_0^{2\pi} [\delta_- (\delta_-^\circ)^*]_{r=\Re} \, d\theta = 0$. So there are the normalization relations for the edge functions $\int_0^{2\pi} [\delta_+ (\delta_+^\circ)^*]_{r=\Re} \, d\theta \equiv (N_\theta)_+$ and $\int_0^{2\pi} [\delta_- (\delta_-^\circ)^*]_{r=\Re} \, d\theta \equiv (N_\theta)_-$, where $(N_\theta)_+$ and $(N_\theta)_-$ are real quantities. For a given direction of a bias magnetic field, the waves described by functions δ_+ and δ_+° propagate only in one direction along the edge. Also the waves described by functions δ_- and δ_-° propagate in one direction (opposite to the former case) along the edge.

It is necessary to point out that formally one can suppose that together with equation (41) the following relation takes place: $(i\mu_a)^\circ = -i\mu_a$. So such orthogonality conditions as $(q_\pm + q_\mp^\circ) \int_0^{2\pi} [\delta_\pm (\delta_\mp^\circ)^*]_{r=\Re} \, d\theta = 0$ may occur. These conditions, however, are beyond a physical meaning. Based on such conditions one cannot restore single valuedness (and, therefore, Hermicity) of the spectral problem. Any observation of the vector-potential terms, the gauge fields and the fluxes (considered below) is excluded in this case.

4. The Aharonov–Bohm-like effect for dipolar-mode magnetization motion in a ferrite disc

The vector potential is considered to be nonobservable in Maxwellian electromagnetism. At the same time, the vector potential can be observable in the Aharonov–Bohm [37] or Aharonov–Casher [38] effects, but only via its line integral, not pointwise. The above analysis shows that a line integral $\oint_C \left(i \frac{\partial \delta_\pm}{\partial \theta} \right) (\delta_\pm^\circ)^* \, dC = \Re \int_0^{2\pi} \left[\left(i \frac{\partial \delta_\pm}{\partial \theta} \right) (\delta_\pm^\circ)^* \right]_{r=\Re} \, d\theta$ is an observable quantity. It follows from the fact that because of such a quantity one can restore single valuedness (and, therefore, Hermicity) of the spectral problem. We can represent this observable quantity as a linear integral of a certain vector potential.

Let us consider an integral $\int_0^{2\pi} \int_S (i\vec{\nabla}_\theta \tilde{\varphi}_p)(\tilde{\varphi}_p^\circ)^* \, dS \, d\theta$, which corresponds to certain mode p. This integral, being a real quantity because of the normalization conditions, can be represented as (we omitted now index p):

$$\int_0^{2\pi} \int_S (i\vec{\nabla}_\theta \tilde{\varphi})(\tilde{\varphi}^\circ)^* \, dS \, d\theta = \int_0^{2\pi} \int_S [\delta_\pm (\delta_\pm^\circ)^*][(i\vec{\nabla}_\theta \tilde{\eta})(\tilde{\eta}^\circ)^*] \, dS \, d\theta$$

$$+ \int_0^{2\pi} \int_S [\tilde{\eta}(\tilde{\eta}^\circ)^*][(i\vec{\nabla}_\theta \delta_\pm)(\delta_\pm^\circ)^*] \, dS \, d\theta. \tag{43}$$

The first integral on the right-hand side (RHS) of equation (43) is the dynamical phase factor for an oscillating mode, but the second integral on the RHS of this equation is the geometrical phase factor. The geometrical phase factor is not single-valued under continuation around a circuit and can be correlated with the vector potential [39]:

$$i\Re \int_0^{2\pi} [(\vec{\nabla}_\theta \delta_\pm)(\delta_\pm^\circ)^*]_{r=\Re} \, d\theta \equiv \oint_C \left(\vec{A}_\theta^m\right)_\pm \cdot d\vec{C} = 2\pi q_\pm. \tag{44}$$

The Berry's phase is generated from the broken dynamical symmetry.

The confinement effect for magnetic-dipolar oscillations requires proper phase relationships to guarantee single valuedness of the wavefunctions. To compensate for sign ambiguities and thus to make wavefunctions single valued we added a vector-potential-type term to the MS-potential Hamiltonian. A circulation of vector $\vec{A}_\theta^m$ should enclose a certain flux. The corresponding flux of the pseudo-electric field $\vec{\in}$ (the gauge field) through a circle of radius $\Re$ is obtained as

$$\int_S (\vec{\in})_\pm \cdot d\vec{S} = \oint_C \left(\vec{A}_\theta^m\right)_\pm \cdot d\vec{C} = (\Xi^e)_\pm, \tag{45}$$

where $(\Xi^e)_\pm$ is the flux of the pseudo-electric field. There should be the positive and negative fluxes. These different-sign fluxes should be inequivalent to avoid the cancellation. This is the case of chirality: the chirality breaks time-reversal symmetry and the positive and negative vortices are not equivalent. Our analysis shows that only the phase factor $\exp[i(\Xi^e)_\pm]$ and not the phase $(\Xi^e)_\pm$ is physically meaningful. An examination of the AB experiment indicates the same fact [40]. Superscript 'm' in the vector-potential term means that there is the physical quantity associated with the magnetization motion or the magnetic-current vector potential (see the appendix). In our problem, the vector potential A_θ^m is due to a surface magnetic current. Because of an irrotational flow on a border contour C, one has $\vec{\nabla} \times \vec{A}_\theta^m = 0$. It means that an electric field on contour C is equal to zero. But the vector potential can be observable via its line integral. A magnetic moment moving along contour C in the gauge field feels no force and undergoes the Aharonov–Bohm-type interference effect.

5. Edge magnetic currents, anapole moments and the magnetoelectric energy splitting in a ferrite disc particle

The most basic implication of the existence of a phase factor in $\tilde{\varphi}$ is operative in the case of the border ring region. The single valuedness of mode membrane functions requires that this phase factor returns to itself modulo 2π on going twice around the circuit. The solutions for functions $\tilde{\varphi}$ depend on both a sign of functions δ and a sign of μ_a. So one can distinguish formally *four solutions* for functions $\tilde{\varphi}$. If the gauge field is viewed as a part of the internal dynamics of the system then the four solutions should correspond four states in a single-Hilbert space.

The above four solutions arise from four functions: $\delta_\pm$ and $\delta_\pm^\circ$. Because of the preferential precession direction for a given direction of a bias field, condition (41) should be taken into account. Therefore, for a given direction of a bias field there are two edge modes: one corresponding to the $(+)$ resonance (when $\delta = \delta_+ = \delta_+^\circ$) and another corresponding to the $(-)$ resonance (when $\delta = \delta_- = \delta_-^\circ$). The topological effects are manifested through the generation of relative phases which accumulate on the boundary wavefunction δ.

In the spectral problem under consideration, the boundary relations are the edge states which are separate from the cross-sectional states. As we discussed above, the difference between the boundary conditions for self-adjointness of operator $\hat{G}_\perp$ and self-adjointness of operator $\hat{L}_\perp$ is due to a non-zero border term on the right-hand side of equation (34):

$$\text{border term} \equiv -\mathrm{i}\mu_a (H_\theta)_{r=\Re}. \tag{46}$$

It will be shown below that this border term can be observable via its circulation integral.

Taking into account equations (2) and (38), we can represent an annual magnetic field H_θ in this term as follows:

$$((\vec{H}_\theta(z))_\pm)_{r=\Re} = -\xi(z)\vec{\nabla}_\theta \tilde{\varphi}_\pm = -\xi(z)(\tilde{\eta}\vec{\nabla}_\theta \delta_\pm + \delta_\pm \vec{\nabla}_\theta \tilde{\eta}), \tag{47}$$

where function $\xi(z)$ characterizes the z-distribution of the MS potential in a ferrite disc [28]. It is evident that the circulation of gradient

$$\vec{\nabla}_\theta \delta_\pm = \frac{1}{\Re}\left.\frac{\partial\delta_\pm}{\partial\theta}\right|_{r=\Re}\vec{e}_\theta = -\mathrm{i}\frac{q_\pm f_\pm}{\Re}\mathrm{e}^{-\mathrm{i}q_\pm\theta}\vec{e}_\theta \tag{48}$$

along contour C gives a non-zero quantity when $q_\pm$ is a number divisible by $\frac{1}{2}$. In contrast, the circulation of gradient $\vec{\nabla}_\theta \tilde{\eta}$ along contour C is equal to zero. We consider the quantity $\nabla_\theta \delta_\pm$ as the velocity of an irrotational 'border' flow:

$$(\vec{V}_\theta)_\pm \equiv \vec{\nabla}_\theta \delta_\pm. \tag{49}$$

In such a sense, functions $\delta_\pm$ are the velocity potentials [41].

The velocity field arises from the Berry phase of the spin field. Figure 3(a) corresponds to the (+) resonance with velocity $(\vec{V}_\theta)_+$ of a chiral rotation and figure 3(b) corresponds to the (−) resonance with velocity $(\vec{V}_\theta)_-$ of a chiral rotation. The non-zero circulation of the velocity $(\vec{V}_\theta)_\pm$ along contour C—the border ring region of a ferrite disc—may have a physical meaning of vorticity [41]. Two cases of chiral rotations described by vectors $(\vec{V}_\theta)_\pm$ can be characterized by vectors of angular moments directed normally to a disc. For a given cross-sectional state (a cross-sectional mode described by membrane function $\tilde{\eta}$), we define the strength of a vortex of a *whole disc* as

$$s_\pm^e \equiv \tilde{\eta}\int_0^d \xi(z)\,\mathrm{d}z \oint_C (\vec{V}_\theta)_\pm \cdot \mathrm{d}\vec{C} = \Re\tilde{\eta}\int_0^d \xi(z)\,\mathrm{d}z \int_0^{2\pi} \nabla_\theta\delta_\pm\,\mathrm{d}\theta = -2f_\pm\tilde{\eta}\int_0^d \xi(z)\,\mathrm{d}z, \tag{50}$$

where d is a disc thickness. A physical meaning of a superscript 'e' in a designation of $\vec{s}_\pm^e$ will be explained below.

We define now an angular moment $\vec{a}_\pm^e$:

$$a_\pm^e \equiv \int_0^d \oint_C (\text{border term})\vec{e}_\theta \cdot \mathrm{d}\vec{C}\,\mathrm{d}z = \mathrm{i}\mu_a s_\pm^e. \tag{51}$$

This angular moment can be formally represented as a result of a circulation of a quantity, which we call a density of an effective boundary magnetic current $\vec{i}^m$:

$$a_\pm^e = 4\pi\int_0^d \oint_C \vec{i}_\pm^m \cdot \mathrm{d}\vec{C}\,\mathrm{d}z, \tag{52}$$

where

$$\vec{i}_\pm^m \equiv \rho^m(\vec{V}_\theta)_\pm \tag{53}$$

and

$$\rho^m \equiv \mathrm{i}\frac{\mu_a}{4\pi}\xi\tilde{\eta}. \tag{54}$$

Relation (52) can be considered as a generalization of the Ampere hypothesis.

The current $\vec{i}^m$ is a persistent magnetic current in an Aharonov–Bohm-like geometry. On an edge ring region, we have a magnetization motion pierced by an electric flux $(\Xi^e)_\pm$. In contrast to the spin-transport analysis of persistent spin current in a mesoscopic ferromagnetic Heisenberg ring in a uniform magnetic field [42], in our continuous-medium model a character

of the magnetization motion becomes apparent via the gyration parameter μ_a in the boundary term for the spectral problem (see equation (36)). There is magnetization motion through a non-simply-connected region. On the edge region, the chiral symmetry of the magnetization precession is broken to form a flux-closure structure. The edge magnetic currents can be observable only via its circulation integrals, not pointwise. This results in the moments oriented along a disc normal. It was shown experimentally [43] that such a moment has a response in an external RF electric field and so can be called as an electric moment. This clarifies a physical meaning of a superscript 'e' in designations of $s_\pm^e$ and $a_\pm^e$.
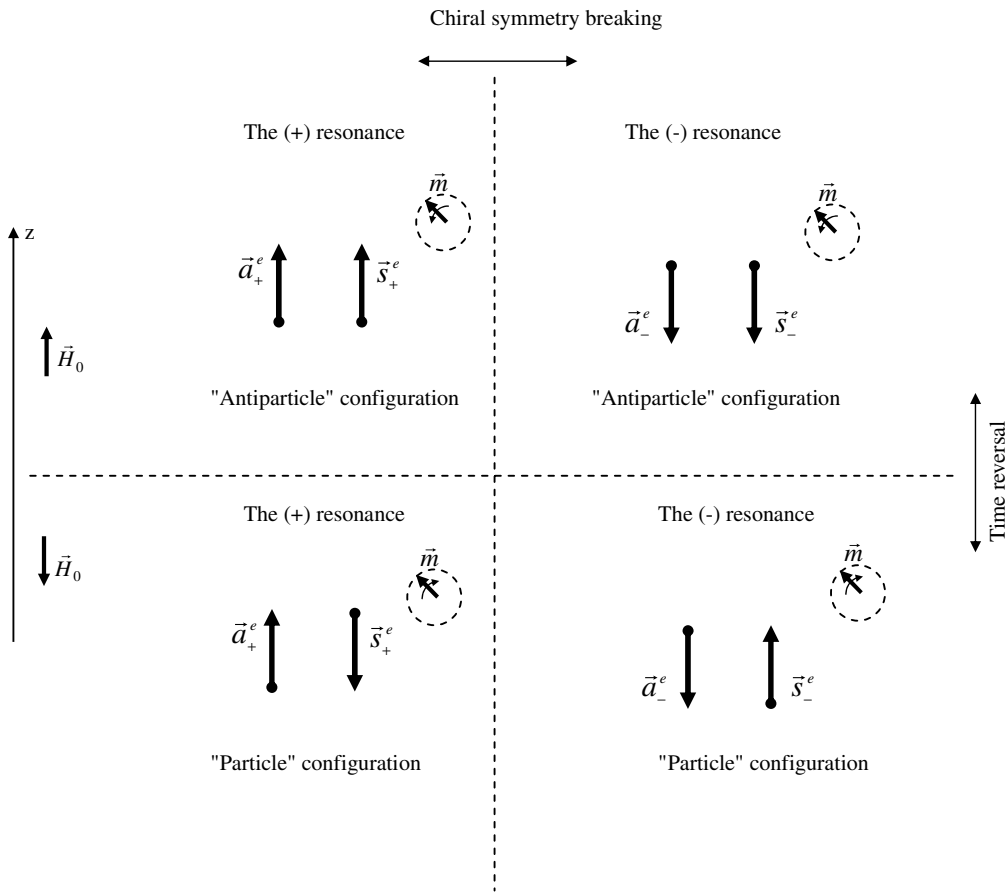
Let us analyse mutual correlations of vectors $\vec{a}^e$ and $\vec{s}^e$ for a ferrite disc particle. These vectors are directed along the $+z$- or $-z$-axis. An analysis of the time-reversal and space-inversion properties of these configurations will clarify the symmetry properties of the edge states. It is evident that for a given type of a resonance (the (+) or (−) resonance), vectors $\vec{a}^e$ and $\vec{s}^e$ can be mutually parallel or antiparallel. These two cases are equally possible. We will conventionally call the case of parallel vectors $\vec{a}^e$ and $\vec{s}^e$ the 'antiparticle' configuration, while the case of antiparallel vectors $\vec{a}^e$ and $\vec{s}^e$ the 'particle' configuration.

Let a bias magnetic field be directed along the $+z$-axis. For the (+) resonance we have ector $\vec{s}_+^e$ directed along the $+z$-axis, while for the (−) resonance we have vector $\vec{s}_-^e$ directed along the $-z$-axis. Suppose that in both cases we have the 'antiparticle' configurations (see figure 4) figure 4. Because of the time-reversal process, the (+) resonance cannot be transformed to the (−) resonance and vice versa. This results in a fact that the direction of vector $\vec{a}^e$ is invariant under the time reversal. So for the time-reversal process shown in figure 4, one has transformations of the 'antiparticle' configurations to the 'particle' configurations.

In a ferrite disc particle, the vector $\vec{a}^e$ can be characterized as an electric moment with the anapole-moment properties. If we classically picture an element of the magnetic current i^m as a small electric-current loop, the combination of all elements along a disc border contour C can be viewed as a toroidal electric-current winding [44, 45]. One can trace a clear analogy between the properties of our structure and the electromagnetic properties of a toroidal solenoid. Following Zel'dovich, such a system has a parity-odd toroidal (or anapole) moment [44, 45]. The toroid dipole (being originated from a toroidally wound solenoid shrunken to a point) singles out a direction in space exactly as does the usual electric dipole. Because of the parity violation, vectors $\vec{a}^e$ become directed in an opposite way for the (+) and (−) resonances. Figure 5 shows transformations of the 'antiparticle' configurations to the 'particle' configurations due to the space-inversion process for the (+) and (−) resonances. For a certain resonance (the (+) resonance or the (−) resonance), an analysis of the PT symmetry of the pictures in figures 4 and 5 shows invariance under such a combined operation. This gives another proof of the Hermicity of the spectral problem [46]. The PT invariance allows excitation of the edge persistent magnetic current in the absence of the external electric flux.

It is evident that a complex conjugate 'particle' configuration is an 'antiparticle' configuration and vice versa. This fact is related to the existence of only a single edge-mode excitation for each wavenumber q. In other words, the 'particle' configurations are their own 'antiparticle' configurations. This resembles the well-known properties of the Majorana fermions [32]. The edge wave propagates only in one direction along the cylindrical surface of a disc. So we have a 'chiral Majorana fermion field' [47, 48] on the lateral wall of a disc.

The demand for single valuedness in the above treatment has been made for the total wavefunction describing a closed system, i.e. a system for which the total Hamiltonian is conserved. However, for scattering of electromagnetic fields by a thin-ferrite-disc particle we

Chiral symmetry breaking

The (+) resonance

$\vec{m}$

$\vec{a}^e_+$ $\vec{s}^e_+$

z

$\vec{H}_0$

"Antiparticle" configuration

The (-) resonance

$\vec{m}$

$\vec{a}^e_-$ $\vec{s}^e_-$

"Antiparticle" configuration

Time reversal

The (+) resonance

$\vec{H}_0$

$\vec{m}$

$\vec{a}^e_+$ $\vec{s}^e_+$

"Particle" configuration

The (-) resonance

$\vec{m}$

$\vec{a}^e_-$ $\vec{s}^e_-$

"Particle" configuration

Figure 4. Time-reversal properties of configurations of vectors $\vec{a}^e$ and $\vec{s}^e$ for the (+) and (−) resonances in correlation with directions of the RF magnetization $\vec{m}$ evolution in a ferrite disc.

have a Hamiltonian which is a function of time. For such an open system, there cannot be the requirement of single valuedness. If the gauge field (a flux) is viewed as external, it should be fixed as a part of the definition of the problem and there will be different energy states for the possible choices of boundary conditions. What picture should appear when one considers scattering of electromagnetic fields by a thin-ferrite-disc particle? For a certain energy level of mode p corresponding to a closed system (see equation (15)), there should be positive and negative splits of energy (with respect to an energy level of mode p) for the 'particle' and 'antiparticle' configurations. The solutions with the positive and negative energy splits are the creation and annihilation operators for the same p-mode energy level. Functions $\tilde{\varphi}$ are vortex complex fields. In such a sense, $\tilde{\varphi}$ and $\tilde{\varphi}^*$ can be viewed as vortex quantum field operators, which destroy and create vortices. We can see that the creation and annihilation operators are operators of the same p-mode light magnon. From the above analysis it follows that different orientations of an electric moment $\vec{a}^e$ (parallel or antiparallel with respect to $\vec{H}_0$) correspond to different energy levels. In an open system, the energy splitting between two cases, $\vec{a}^e \cdot \vec{H}_0 > 0$ and $\vec{a}^e \cdot \vec{H}_0 < 0$, can be characterized as the *magnetoelectric* energy splitting.
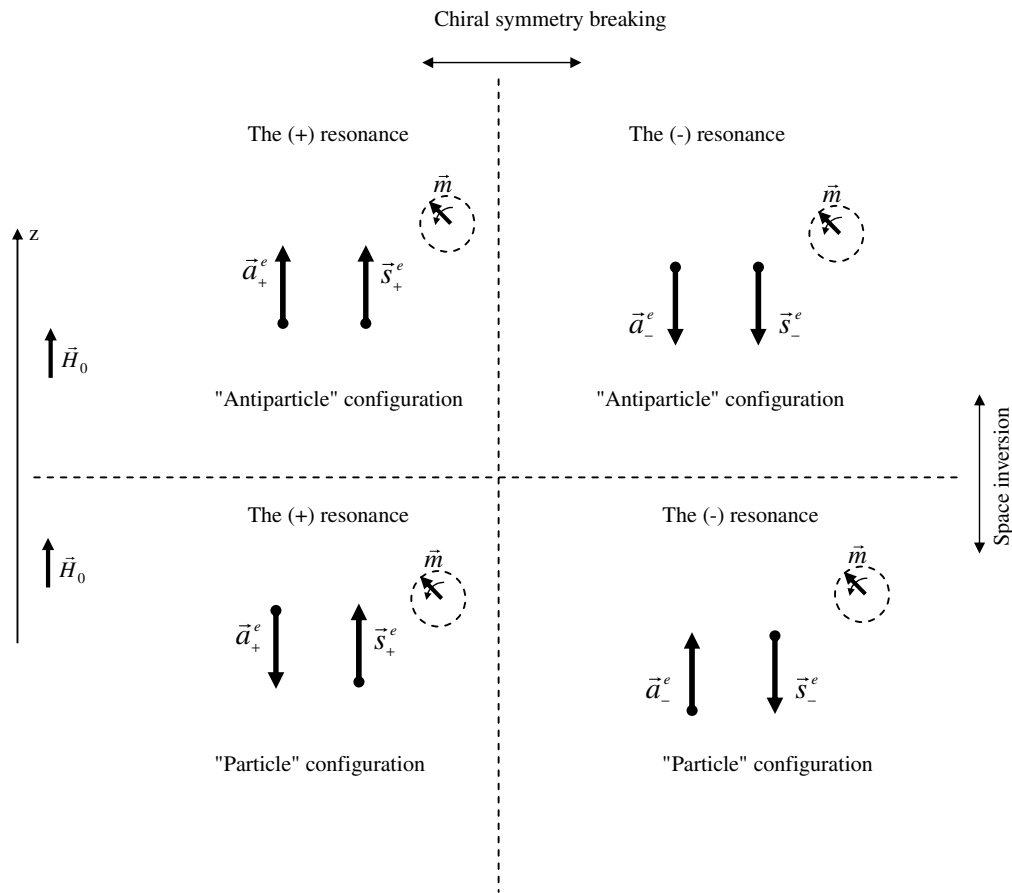
Figure 5. Space-inversion properties of configurations of vectors $\vec{a}^{\,e}$ and $\vec{s}^{\,e}$ for the (+) and (−) resonances in correlation with directions of the RF magnetization $\vec{m}$ evolution in a ferrite disc.

6. Discussion and conclusion

In ferrite discs, the microwave vortices can appear in different kinds of physical phenomena depending on space scales of the wave processes. On the one hand, one can observe the exchange-interaction vortices of magnetization [16, 21]. On the other hand, there are the electromagnetic-wave (Poynting-vector) vortices [6]. Space scales of the MS-wave processes are much smaller than the wavelengths of electromagnetic waves and much higher than the wavelengths of exchange waves. Physically, the magnetic-dipolar-mode oscillations are neither electromagnetic-wave nor exchange-wave oscillations [27, 30, 34]. In this paper, we showed unique features of the MS-mode vortices.

The confinement effect for magnetic-dipolar oscillations requires proper phase relationships to guarantee single valuedness of the wavefunctions. To compensate for sign ambiguities and thus to make wavefunctions single valued we added a vector-potential-type term to the MS-potential Hamiltonian. This resembles the procedure made by Mead for the Born–Oppenheimer wavefunctions in molecular systems [49–51]. The flux of the

pseudo-electric field $\vec{\in}$ (the gauge field) through a circle of radius $\Re$ (see equation (45)) is similar to the flux of the pseudo-magnetic field obtained in [49, 50] for molecular systems.

A combined phenomenon of the breaking of spin-rotation and spatial-rotation symmetries is well known in a theory of a Bose–Einstein condensate [47, 48]. For magnetic-dipolar modes in a normally magnetized thin-ferrite disc, the border–region interaction can be interpreted in terms of a chiral contribution to the precessing-electron magnetization. In order to cancel non-single valuedness of the magnetic-dipolar modes, the boundary excitations must be described by chiral states. The chiral states appear as rings of vortices moving around the apparently undisturbed membrane-function discrete-energy states. The chiral edge states constitute a subset of the spectrum. A feature of the Dirac-like nature of the chiral edge spectrum is that there are two kinds of modes with the 'particle' and 'antiparticle' configurations. This can be described by saying that there are half-integer quantum numbers of the vortices ('charges') and that 'particle' and 'antiparticle' configurations carry opposite values of these quantum numbers. This is the case of dynamical symmetry breaking. The chirality breaks time-reversal symmetry, and the positive and negative vortices are not equivalent. Appropriate quasiparticles are just effective fields describing the collective excitations on the edge and are considered in terms of which a complicated strongly interacting system appears weakly interacting.

The gauge field (fluxes) can be viewed either as external or as a part of the internal dynamics of the system. The demand for single valuedness in our treatment has been made for the total MS-potential function describing a closed system, i.e. a stationary-state system for which the total Hamiltonian is conserved. For an open system—the system interacting with an external electromagnetic field—we do not have the requirement of single valuedness. Such edge states, existing in the thin-walled cylinder and having a certain handedness, can be excited only by increasing (decreasing) the angular moment of a whole ferrite disc [43]. Together with interaction with the magnetic component of the electromagnetic radiation, a small magnetic disc interacts with the electric component of the electromagnetic field. This is because of dynamical symmetry properties of magnetostatic modes. The dynamical symmetry breaking in quasistatic magnetic oscillations shows special-type gauge transformation for the fields.

Appendix. Electric-current and magnetic-current vector potentials

In a case of the 'true' AB effect we have the electric-current vector potential resulting, in fact, from the electric-charge motion. In a case of the MDM AB-like effect we have the magnetic-current vector potential resulting from the magnetization motion. The 'true' AB effect comes from the gauge-invariant coupling between the electric current and the electric-current vector potential. For that reason this effect can be observed even in the absence of the magnetic field. The MDM AB-type interference comes from the gauge-invariant coupling between the 'magnetic' current and the magnetic-current vector potential.

In the electromagnetic field theory, the motion equations of a charge in the electromagnetic field give correlations between the $\vec{E}$ and $\vec{B}$ fields and the vector $\vec{A}$ and scalar φ potentials:

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\varphi; \qquad \vec{B} = \nabla \times \vec{A}. \tag{A.1}$$

The same relations one immediately obtains from the Maxwell equations. For the electric current density $\vec{j}^e$, as a source of the electromagnetic field, one has the wave equation for

vector potential $\vec{A}$ [17]:

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} \vec{j}^e. \tag{A.2}$$

In spite of the fact that no magnetic charges and no motion equations for magnetic charges are known in nature, because of the electromagnetic duality one can formally introduce magnetic currents in Maxwell equations. This formal procedure allows solving numerous electrodynamics problems [52, 53]. To distinguish an electrodynamic vector potential caused by an electric current from an electrodynamic vector potential caused by a magnetic current, we rewrite equation (A.2) as

$$\nabla^2 \vec{A}^e - \frac{1}{c^2} \frac{\partial^2 \vec{A}^e}{\partial t^2} = -\frac{4\pi}{c} \vec{j}^e, \tag{A.3}$$

where $\vec{A}^e$ means the electric-current vector potential. Together with this wave equation one has from Maxwell equations another wave equation [52, 54]

$$\nabla^2 \vec{A}^m - \frac{1}{c^2} \frac{\partial^2 \vec{A}^m}{\partial t^2} = -\frac{4\pi}{c} \vec{j}^m, \tag{A.4}$$

where $\vec{A}^m$ means the magnetic-current vector potential.

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